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Similarity solutions for the two-dimensional non-stationary ideal MHD equations

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Abstract. A similarity analysis of the non-linear two-dimensional non-stationary ideal MHD equations is presented. In the case of a magnetic field perpendicular to the isentropic motion of the plasma, we establish the complete Lie algebra of infinitesimal symmetries. The laws of conservation are mentioned. The similarity method for partial differential equations as a procedure for reducing the number of independent variables is applied repeatedly. Finally we obtain systems of ordinary differential equations for similarity solutions of the MHD equations considered.

1. Introduction

The one-fluid theory is a well known starting point for studies of the time evolution of plasmas from a macroscopic point of view. For long spatial scale and low-frequency phenomena in plasmas, a simplified set of the one-fluid equations and Maxwell's equations has been developed, called magnetohydrodynamic (MHD) equations (Hughes and Young 1966). If dissipative effects are neglected, the MHD theory is called ideal.

In this paper we carry out a similarity analysis of the non-linear two-dimensional non-stationary ideal MHD equations in the case of a magnetic field perpendicular to the isentropic motion of the plasma. The similarity method of the analysis of partial differential equations is well described in the literature (Bluman and Cole 1974, Ovsiannikov 1982). Therefore, we omit the known details of the procedures. Our notation and terminology are similar to that of Bluman and Cole (1974).

Lie group analysis for non-stationary MHD equations has been used to study incompressible fluids (Nucci 1984) and compressible one-dimensional plasmas (Groß 1983).

In what follows, we determine the full Lie symmetry group of a system of non-stationary two-dimensional MHD equations for compressible plasmas. Turning to the laws of conservation, we mention that, in the case of an adiabatic exponent $\gamma = 2$, there are two additional conservation laws. Solutions which are themselves invariant under some subgroup of the symmetry group are called similarity solutions. They can all be found by solving a system of partial differential equations with only two independent (similarity) variables. We discuss some different cases of these reduced equations, especially with regard to boundary or initial value problems. In four cases, we apply once more the procedure for reduction and obtain systems of ordinary differential equations which may be solved numerically by standard procedures.

2. Lie symmetries of the two-dimensional non-stationary ideal MHD equations

The system we want to study is the set of non-stationary ideal MHD equations in the case of a plane isentropic motion of the plasma across a magnetic field. Let x, y, z be cartesian coordinates and $\boldsymbol{e}_x$, $\boldsymbol{e}_y$, $\boldsymbol{e}_z$ the corresponding unit vectors. All variables being written as dimensionless quantities, the appropriate equations are

$$\frac{\partial \rho}{\partial t}+\rho\left(\frac{\partial v^x}{\partial x}+\frac{\partial v^y}{\partial y}\right)+v^x\frac{\partial \rho}{\partial x}+v^y\frac{\partial \rho}{\partial y}=0 \tag{1a}$$

$$\rho\left(\frac{\partial v^x}{\partial t}+v^x\frac{\partial v^x}{\partial x}+v^y\frac{\partial v^x}{\partial y}\right)+R_H h\frac{\partial h}{\partial x}+\frac{\partial p}{\partial x}=0 \tag{1b}$$

$$\rho\left(\frac{\partial v^y}{\partial t}+v^x\frac{\partial v^y}{\partial x}+v^y\frac{\partial v^y}{\partial y}\right)+R_H h\frac{\partial h}{\partial y}+\frac{\partial p}{\partial y}=0 \tag{1c}$$

$$\frac{\partial h}{\partial t}+h\left(\frac{\partial v^x}{\partial x}+\frac{\partial v^y}{\partial y}\right)+v^x\frac{\partial h}{\partial x}+v^y\frac{\partial h}{\partial y}=0 \tag{1d}$$

$$\frac{\partial p}{\partial t}+\gamma p\left(\frac{\partial v^x}{\partial x}+\frac{\partial v^y}{\partial y}\right)+v^x\frac{\partial p}{\partial x}+v^y\frac{\partial p}{\partial y}=0 \tag{1e}$$

where $\boldsymbol{v}=v^x(x,y,t)\boldsymbol{e}_x+v^y(x,y,t)\boldsymbol{e}_y$ is the fluid velocity, $\boldsymbol{H}=h(x,y,t)\boldsymbol{e}_z$ is the magnetic field, $\rho=\rho(x,y,t)$ is the mass density, $p=p(x,y,t)$ is the pressure, γ is the adiabatic exponent and $R_H=\mu_0 H_0^2/\rho_0 U_0^2$ is the magnetic pressure number (where μ_0 is the magnetic permeability in free space and H_0, ρ_0, U_0 are some reference values of the magnetic field, the mass density and the velocity). Equations (1) are a non-linear system of first-order partial differential equations with three independent and five dependent variables.

In the spirit of Lie we consider for x, y, t, v^x, v^y, h, ρ and p infinitesimal transformations of the form

$$\begin{aligned}&\tilde{x}=x+\varepsilon\xi^x(x,y,t,v^x,v^y,h,\rho,p)+\mathrm{O}(\varepsilon^2)\\&\qquad\text{(analogous transformations for } y, t)\\&\tilde{v}^x=v^x+\varepsilon\eta^{v^x}(x,y,t,v^x,v^y,h,\rho,p)+\mathrm{O}(\varepsilon^2)\\&\qquad\text{(analogous transformations for } v^y, h, \rho, p)\end{aligned} \tag{2}$$

which leave (1) invariant.

The determining equations for the infinitesimals ξ^x, ξ^y, ξ^t, η^{v^x}, η^{v^y}, η^h, η^ρ and η^p are obtained by following the procedure given in Bluman and Cole (1974). This results in a linear homogeneous system of partial differential equations for ξ^x, ξ^y, ξ^t, η^{v^x}, η^{v^y}, η^h, η^ρ and η^p with 216 equations which can be simplified by linear combinations of suitable equations.

The solution for the determining equations contains nine arbitrary parameters $C_1,\ldots,C_9$ in the case of $\gamma\neq 2$, whereas for $\gamma=2$ there are ten arbitrary parameters $C_1,\ldots,C_{10}$ and an arbitrary function $f(u,w)$, where $u=h/\rho$ and $w=p/\rho^2$. With these parameters and this function we find the infinitesimals for the case of $\gamma = 2$ to be

$$\begin{aligned}
\xi^x &= C_1 + C_4 t + C_6 y + C_7 x + C_{10} x t \\
\xi^y &= C_2 + C_5 t - C_6 x + C_7 y + C_{10} y t \\
\xi^t &= C_3 + C_8 t + C_{10} t^2 \\
\eta^{v^x} &= C_4 + C_6 v^y + (C_7 - C_8) v^x + C_{10}(x - v^x t) \\
\eta^{v^y} &= C_5 - C_6 v^x + (C_7 - C_8) v^y + C_{10}(y - v^y t) \\
\eta^h &= C_9 h - 2C_{10} h t + \rho f(u, w) \\
\eta^\rho &= 2(C_9 + C_8 - C_7)\rho - 2C_{10}\rho t \\
\eta^p &= 2C_9 p - 4C_{10} p t - R_H h \rho f(u, w).
\end{aligned} \tag{3}$$

The case of $\gamma \neq 2$ can be obtained from the case of $\gamma = 2$ by setting $C_{10} = 0$ and $f \equiv 0$.

The set of the infinitesimal operators

$$Z = \xi^x \frac{\partial}{\partial x} + \xi^y \frac{\partial}{\partial y} + \xi^t \frac{\partial}{\partial t} + \eta^{v^x} \frac{\partial}{\partial v^x} + \eta^{v^y} \frac{\partial}{\partial v^y} + \eta^h \frac{\partial}{\partial h} + \eta^\rho \frac{\partial}{\partial \rho} + \eta^p \frac{\partial}{\partial p} \tag{4}$$

with the infinitesimals (3) forms a Lie algebra with the commutator $[Z_\alpha, Z_\beta] = Z_\alpha Z_\beta - Z_\beta Z_\alpha$. In the case of $\gamma = 2$ an infinite-dimensional Lie algebra is obtained, whereas for $\gamma \neq 2$ we obtain a nine-dimensional Lie algebra which is a sub-algebra of the algebra associated with $\gamma = 2$. A basis of the Lie algebra is obtained by setting one of the parameters $C_1, \ldots, C_{10}$ in (4) equal to one, while all other parameters and the free function f are equated to zero, or by setting $C_1 = \ldots = C_{10} = 0$ and using the function f not equal to zero:

$$\begin{aligned}
&X_1 = \partial/\partial x \qquad X_2 = \partial/\partial y \qquad X_3 = \partial/\partial t \qquad X_4 = \frac{\partial}{\partial v^x} + t\frac{\partial}{\partial x} \qquad X_5 = \frac{\partial}{\partial v^y} + t\frac{\partial}{\partial y} \\
&X_6 = v^y \frac{\partial}{\partial v^x} - v^x \frac{\partial}{\partial v^y} + y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} \qquad X_7 = v^x \frac{\partial}{\partial v^x} + v^y \frac{\partial}{\partial v^y} - 2\rho\frac{\partial}{\partial \rho} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} \\
&X_8 = -v^x \frac{\partial}{\partial v^x} - v^y \frac{\partial}{\partial v^y} + 2\rho\frac{\partial}{\partial \rho} + t\frac{\partial}{\partial t} \qquad X_9 = h\frac{\partial}{\partial h} + 2\rho\frac{\partial}{\partial \rho} + 2p\frac{\partial}{\partial p} \\
&X_{10} = xt\frac{\partial}{\partial x} + yt\frac{\partial}{\partial y} + t^2\frac{\partial}{\partial t} + (x - tv^x)\frac{\partial}{\partial v^x} + (y - tv^y)\frac{\partial}{\partial v^y} - 2th\frac{\partial}{\partial h} - 2t\rho\frac{\partial}{\partial \rho} - 4tp\frac{\partial}{\partial p} \\
&X(f) = f(u, w)\left(\rho\frac{\partial}{\partial h} - R_H h \rho \frac{\partial}{\partial p}\right).
\end{aligned} \tag{5}$$

Here the numbering of the base vectors corresponds to that of the parameters $C_1, \ldots, C_{10}$, and $X(f)$ is the vector corresponding to the free function f. In the case of $\gamma \neq 2$ the base vectors are $X_1, \ldots, X_9$, whereas for $\gamma = 2$, X_{10} and $X(f)$ are added. The commutator table of the Lie algebra is given in table 1.

Since the infinitesimals ξ^x, ξ^y, ξ^t, η^{v^x}, η^{v^y}, η^h, η^ρ and η^p are known, the associated global transformations can be obtained by solving the system of ordinary differential equations

$$\begin{aligned}
&\mathrm{d}\tilde{x}/\mathrm{d}\varepsilon = \xi^x(\tilde{x}, \tilde{y}, \tilde{t}, \tilde{v}^x, \tilde{v}^y, \tilde{h}, \tilde{\rho}, \tilde{p}) \qquad \tilde{x}(\varepsilon = 0) = x \\
&\qquad \text{(analogous equations for } \tilde{y}, \tilde{t}) \\
&\mathrm{d}\tilde{v}^x/\mathrm{d}\varepsilon = \eta^{v^x}(\tilde{x}, \tilde{y}, \tilde{t}, \tilde{v}^x, \tilde{v}^y, \tilde{h}, \tilde{\rho}, \tilde{p}) \qquad \tilde{v}^x(\varepsilon = 0) = v^x \\
&\qquad \text{(analogous equations for } \tilde{v}^y, \tilde{h}, \tilde{\rho}, \tilde{p}).
\end{aligned} \tag{6}$$

Table 1. Commutators of the Lie algebra given by (5).

	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	X_9	X_{10}	$X(f)$
X_1	0	0	0	0	0	$-X_2$	X_1	0	0	X_4	0
X_2	0	0	0	0	0	X_1	X_2	0	0	X_5	0
X_3	0	0	0	X_1	X_2	0	0	X_3	0	$2X_8+X_7-2X_9$	0
X_4	0	0	$-X_1$	0	0	$-X_5$	X_4	$-X_4$	0	0	0
X_5	0	0	$-X_2$	0	0	X_4	X_5	$-X_5$	0	0	0
X_6	X_2	$-X_1$	0	X_5	$-X_4$	0	0	0	0	0	0
X_7	$-X_1$	$-X_2$	0	$-X_4$	$-X_5$	0	0	0	0	0	$-2X(f)+2X\left(u\frac{\partial f}{\partial u}\right)+4X\left(n\frac{\partial f}{\partial w}\right)$
X_8	0	0	$-X_3$	X_4	X_5	0	0	0	0	X_{10}	$2X(f)-2X\left(u\frac{\partial f}{\partial u}\right)-4X\left(w\frac{\partial f}{\partial w}\right)$
X_9	0	0	0	0	0	0	0	0	0	0	$X(f)-X\left(u\frac{\partial f}{\partial u}\right)-2X\left(w\frac{\partial f}{\partial w}\right)$
X_{10}	$-X_4$	$-X_5$	$-2X_8-X_7+2X_9$	0	0	0	0	$-X_{10}$	0	0	0
$X(f)$	0	0	0	0	0	0	$2X(f)-2X\left(u\frac{\partial f}{\partial u}\right)-4X\left(w\frac{\partial f}{\partial w}\right)$	$-2X(f)+2X\left(u\frac{\partial f}{\partial u}\right)+4X\left(w\frac{\partial f}{\partial w}\right)$	$-X(f)+X\left(u\frac{\partial f}{\partial u}\right)+2X\left(w\frac{\partial f}{\partial w}\right)$	0	0

In the following, the global one-parameter transformations associated with the base vectors $X_1, \ldots, X_{10}$ and $X(f)$ of the Lie algebra are given. In doing so, only those variables which are changed by the transformation are mentioned:

$C_1 \neq 0$: $\tilde{x} = x + \varepsilon C_1$ (translation in the x direction)

$C_2 \neq 0$: $\tilde{y} = y + \varepsilon C_2$ (translation in the y direction)

$C_3 \neq 0$: $\tilde{t} = t + \varepsilon C_3$ (time translation)

$C_4 \neq 0$: $\tilde{x} = x + \varepsilon C_4 t$ $\quad \tilde{v}^x = v^x + \varepsilon C_4$ (Galilei transformation)

$C_5 \neq 0$: $\tilde{y} = y + \varepsilon C_5 t$ $\quad \tilde{v}^y = v^y + \varepsilon C_5$ (Galilei transformation)

$C_6 \neq 0$: $\tilde{x} = y \sin(C_6\varepsilon) + x \cos(C_6\varepsilon)$ $\quad \tilde{v}^x = v^y \sin(C_6\varepsilon) + v^x \cos(C_6\varepsilon)$

$\tilde{y} = y \cos(C_6\varepsilon) - x \sin(C_6\varepsilon)$ $\quad \tilde{v}^y = v^y \cos(C_6\varepsilon) - v^x \sin(C_6\varepsilon)$

(rotation)

$C_7 \neq 0$: $\tilde{x} = x\, \mathrm{e}^{C_7\varepsilon}$ $\quad \tilde{v}^x = v^x\, \mathrm{e}^{C_7\varepsilon}$ $\quad \tilde{\rho} = \rho\, \mathrm{e}^{-2C_7\varepsilon}$

$\tilde{y} = y\, \mathrm{e}^{C_7\varepsilon}$ $\quad \tilde{v}^y = v^y\, \mathrm{e}^{C_7\varepsilon}$ (scaling for x, y, v^x, v^y, ρ)

$C_8 \neq 0$: $\tilde{t} = t\, \mathrm{e}^{C_8\varepsilon}$ $\quad \tilde{v}^x = v^x\, \mathrm{e}^{-C_8\varepsilon}$ $\quad \tilde{v}^y = v^y\, \mathrm{e}^{-C_8\varepsilon}$ $\quad \tilde{\rho} = \rho\, \mathrm{e}^{2C_8\varepsilon}$

(scaling for t, v^x, v^y, ρ)

$C_9 \neq 0$: $\tilde{h} = h\, \mathrm{e}^{C_9\varepsilon}$ $\quad \tilde{\rho} = \rho\, \mathrm{e}^{2C_9\varepsilon}$ $\quad \tilde{p} = p\, \mathrm{e}^{2C_9\varepsilon}$ (scaling for h, ρ, p).

The following transformations are possible in the case of $\gamma = 2$ only:

$C_{10} \neq 0$: $\tilde{x} = \dfrac{x}{1 - \varepsilon C_{10} t}$ $\quad \tilde{v}^x = v^x + \varepsilon C_{10}(x - v^x t)$

$\tilde{y} = \dfrac{y}{1 - \varepsilon C_{10} t}$ $\quad \tilde{v}^y = v^y + \varepsilon C_{10}(y - v^y t)$

$\tilde{t} = \dfrac{t}{1 - \varepsilon C_{10} t}$ $\quad \tilde{h} = \dfrac{h}{1 + 2\varepsilon C_{10} t}$

$\tilde{\rho} = \dfrac{\rho}{1 + 2\varepsilon C_{10} t}$ $\quad \tilde{p} = \dfrac{p}{1 + 4\varepsilon C_{10} t}$

(projective transformation for x, y, t, h, ρ, p).

$f(u, w) \neq 0$: $\mathrm{d}\tilde{u}/\mathrm{d}\varepsilon = f(\tilde{u}, \tilde{w})$ $\quad \tilde{u}(\varepsilon = 0) = u$ $\quad \tilde{w} = w + \frac{1}{2} R_H (u^2 - \tilde{u}^2)$

i.e. $\tilde{p} + \frac{1}{2} R_H \tilde{h}^2 = p + \frac{1}{2} R_H h^2$.

3. Conservation laws

The local equations of balance $(1a)$–$(1e)$ are partly themselves conservation laws or may be used to establish further conservation laws. For suitably smooth regions and fields, a (local) conservation law takes the form

$$\partial\tau/\partial t + \operatorname{div}(\tau \boldsymbol{v} + \boldsymbol{\zeta}) = 0 \tag{7}$$

in which τ denotes a volume density, whereas $\tau\boldsymbol{v}$ and $\boldsymbol{\zeta}$ denote the convective and non-convective fluxes, respectively.

The equations (1*a*, *d*) are already in the form of a conservation law, with $\zeta = 0$ and $\tau = \rho$, respectively. The momentum equations (1*b*, *c*), coupled with (1*a*), yield the laws of conservation of linear momentum with $\tau = \rho v^i$ and $\zeta = (p + \frac{1}{2}R_H h^2)e_i$, $i = x, y$. If we identify τ as the energy density $\frac{1}{2}\rho|v|^2 + p/(\gamma - 1) + \frac{1}{2}R_H h^2$, the law of conservation of energy is obtained by virtue of (1*a*)-(*e*) with $\zeta = (p + \frac{1}{2}R_H h^2)v$. A coupling of the two equations for the conservation of linear momentum leads to the law of conservation of angular momentum, with $\tau = \rho(xv^y - yv^x)$ and $\zeta = (p + \frac{1}{2}R_H h^2)\ (xe_y - ye_x)$.

In the case of an adiabatic exponent $\gamma = 2$, two additional conservation laws are produced by the method of applying admitted symmetry group generators to known conservation laws (Ibragimov 1985, Olver 1986). According to the discussion of the motion of a perfect polytropic gas by Ibragimov (1985), we select the vector field X_{10} and use the associated vector field

$$\overline{X_{10}} = \left(x - tv^x - xt\frac{\partial v^x}{\partial x} - yt\frac{\partial v^x}{\partial y} - t^2\frac{\partial v^x}{\partial t}\right)\frac{\partial}{\partial v^x} + \left(y - tv^y - xt\frac{\partial v^y}{\partial x} - yt\frac{\partial v^y}{\partial y} - t^2\frac{\partial v^y}{\partial t}\right)\frac{\partial}{\partial v^y}$$
$$- t\left(2\rho + x\frac{\partial \rho}{\partial x} + y\frac{\partial \rho}{\partial y} + t\frac{\partial \rho}{\partial t}\right)\frac{\partial}{\partial \rho} - t\left(4p + x\frac{\partial p}{\partial x} + y\frac{\partial p}{\partial y} + t\frac{\partial p}{\partial t}\right)\frac{\partial}{\partial p} \tag{8}$$
$$- t\left(2h + x\frac{\partial h}{\partial x} + y\frac{\partial h}{\partial y} + t\frac{\partial h}{\partial t}\right)\frac{\partial}{\partial h}.$$

Applying $\overline{X_{10}}$ to the law of conservation of energy, we obtain a conservation law with $\tau = t(\rho|v|^2 + 2p + R_H h^2) - \rho x \cdot v$ and $\zeta = (p + \frac{1}{2}R_H h^2)(2tv - x)$, where $x = xe_x + ye_y$. By applying $\overline{X_{10}}$ to this conservation law, another conservation law with $\tau = t^2(\rho|v|^2 + 2p + R_H h^2) - \rho x \cdot (2tv - x)$ and $\zeta = (2p + R_H h^2)(tv - x)t$ can be constructed.

4. Reduction of independent variables

In order to reduce (1) to a system of partial differential equations with only two independent variables, we construct similarity variables and similarity forms of field variables. According to the well known method (Bluman and Cole 1974) we have to solve the system of characteristic equations

$$\frac{dx}{\xi^x} = \frac{dy}{\xi^y} = \frac{dt}{\xi^t} = \frac{dv^x}{\eta^{v^x}} = \frac{dv^y}{\eta^{v^y}} = \frac{dh}{\eta^h} = \frac{d\rho}{\eta^\rho} = \frac{dp}{\eta^p}. \tag{9}$$

In general, the infinitesimals are given by (3). If particular boundary or initial values are given it is advisable to find out immediately the subgroup of (3) leaving these conditions invariant, provided there exists a similarity solution for the particular problem.

In the following we first discuss solutions of (9) with infinitesimals (3) in the case of $\gamma \neq 2$ (or $\gamma = 2$ and $C_{10} = 0$, $f \equiv 0$). Subsequently we investigate some boundary or initial value problems.

Integration of the first-order differential equations corresponding to pairs of equations involving only independent variables of (9) leads to similarity variables, called λ and μ, which are given as the constants in the solutions. We distinguish two cases:

I:	$\xi^t \neq 0$	$C_3 \neq 0$ or $C_8 \neq 0$
II:	$\xi^t = 0$	$C_3 = C_8 = 0$.

In the first case one obtains a non-homogeneous linear system of ordinary differential equations with constant coefficients in the case $C_8 = 0$. Therefore we distinguish the cases $C_8 \neq 0$ (case I1) and $C_8 = 0$ (case I2). The non-homogeneous system of ordinary differential equations can be easily solved. We find different types of solutions corresponding to the cases

I1a: $C_6 \neq 0$ or $C_6 = 0$, $C_7 \neq 0$, $C_7 \neq C_8$

I1b: $C_6 = 0$, $C_7 = C_8 \neq 0$

I1c: $C_6 = 0$, $C_7 = 0$

I2a: $C_6 \neq 0$ or $C_7 \neq 0$

I2b: $C_6 = 0$, $C_7 = 0$.

In case II, corresponding to $\xi^t = 0$, we have a similarity variable equal to t and the first pair of equations in (9) leads to the distinction

II1: $\xi^x \neq 0$ $\quad$ $C_1 \neq 0$ or $C_4 \neq 0$ or $C_6 \neq 0$ or $C_7 \neq 0$

II2: $\xi^x = 0$, $\xi^y \neq 0$ $\quad$ $C_1 = C_4 = C_6 = C_7 = 0$, $C_2 \neq 0$ or $C_5 \neq 0$.

For $C_1 = C_2 = C_3 = C_4 = C_5 = C_6 = C_7 = C_8 = 0$ we have $\xi^x = \xi^y = \xi^t = 0$ and it can also be shown that $\eta^{v^x} = \eta^{v^y} = \eta^h = \eta^\rho = \eta^p = 0$, i.e. $C_9 = 0$. Hence the transformation (2) is the identical transformation for both independent and dependent variables.

In general, the values of the dependent variables v^x, v^y, h, ρ and p change along the characteristic curves given by $\lambda =$ constant and $\mu =$ constant in the space with coordinates (x, y, t). Corresponding to the cases distinguished, these variations can be found by integrating one of the three systems of characteristic equations

$$\frac{\mathrm{d}x}{\xi^x} = \frac{\mathrm{d}v^x}{\eta^{v^x}} = \frac{\mathrm{d}v^y}{\eta^{v^y}} = \frac{\mathrm{d}h}{\eta^h} = \frac{\mathrm{d}\rho}{\eta^\rho} = \frac{\mathrm{d}p}{\eta^p}$$

$$\frac{\mathrm{d}y}{\xi^y} = \frac{\mathrm{d}v^x}{\eta^{v^x}} = \frac{\mathrm{d}v^y}{\eta^{v^y}} = \frac{\mathrm{d}h}{\eta^h} = \frac{\mathrm{d}\rho}{\eta^\rho} = \frac{\mathrm{d}p}{\eta^p}$$

$$\frac{\mathrm{d}t}{\xi^t} = \frac{\mathrm{d}v^x}{\eta^{v^x}} = \frac{\mathrm{d}v^y}{\eta^{v^y}} = \frac{\mathrm{d}h}{\eta^h} = \frac{\mathrm{d}\rho}{\eta^\rho} = \frac{\mathrm{d}p}{\eta^p}.$$

The solutions contain some 'constants' of integration which are functions of λ and μ. These are the new dependent variables, called $U(\lambda, \mu)$, $V(\lambda, \mu)$, $H(\lambda, \mu)$, $R(\lambda, \mu)$ and $P(\lambda, \mu)$. In any case, substitution of the new dependent variables into (1) leads to a new system of partial differential equations for these variables with only two independent variables, namely λ and μ.

We present the results of this procedure in the appendix. It should be mentioned that our classification does not correspond with an optimal system of subgroups as defined by Ovsiannikov (1982).

So far no restrictions have been made by using prescribed boundary or initial values. Thus let us suppose boundary or initial values $b(x, y, t, v^x, v^y, h, \rho, p) = 0$ on the curves $w(x, y, t) = 0$.

In general we cannot use the whole transformation group leaving (1) invariant to construct similarity solutions which also satisfy the given boundary or initial values, but only a subgroup.

In order to find such subgroups it is sufficient to determine transformations which not only leave the system of differential equations invariant but which also leave the boundary or initial conditions and the curves on which the conditions are prescribed invariant.

In order to determine those transformations which leave the curve $w(x, y, t)=0$ invariant one has to examine if $w(\tilde{x}, \tilde{y}, \tilde{t})=0$ follows whenever $w(x, y, t)=0$. This yields relations between the constants $C_1, \ldots, C_9$, and the number of free parameters is thus reduced. It also follows that not all of the different cases which have been found in order to reduce the system (1) to a system with only two independent variables λ and μ can be used for a special boundary or initial value problem.

In the following we represent the subgroups which leave given curves $w(x, y, t)=0$ invariant, the cases which can be used to reduce the system (1) to a system with two independent variables and the curves $W(\lambda, \mu)=0$ in the $\lambda\mu$ plane on which conditions for the reduced system must be prescribed.

In general, invariance of the given boundary or initial values $b(x, y, t, v^x, v^y, h, \rho, p)=0$ further restricts the transformation groups.

4.1. Boundary values on lines

Let boundary values be prescribed on lines $y=y_0-mx$, y_0, $m\in\mathbb{R}$, i.e. $w(x, y, t)=y-y_0+mx$. The transformations which leave these lines invariant must satisfy

$$C_2=-C_1m-C_7y_0 \qquad C_5=-C_4m \qquad C_6=0.$$

Hence only cases I1a, b, c, I2a, b, II1aβ and II1b can be used to reduce the system (1) to a system with two independent variables.

The curves $W(\lambda, \mu)=0$ on which conditions for the reduced system must be prescribed are

I1a: $W(\lambda, \mu)=\lambda+m\mu$

I1b: $W(\lambda, \mu)=\mu+m\lambda$

I1c: $W(\lambda, \mu)=\mu-y_0+m\lambda$

I2a: $W(\lambda, \mu)=\mu+m\lambda$

I2b: $W(\lambda, \mu)=\mu-y_0+m\lambda$

II1aβ: $W(\lambda, \mu)=\mu+m$

II1b: $W(\lambda, \mu)=\mu-y_0$.

4.2. Boundary values on lines parallel to the y axis

Let boundary values be prescribed on lines $x=x_0$, $x_0\in\mathbb{R}$, i.e. $w(x, y, t)=x-x_0$. The transformations which leave these lines invariant must satisfy

$$C_1=-C_7x_0 \qquad C_4=0 \qquad C_6=0.$$

Hence only cases I1a, b, c, I2a, b and II2 can be used to reduce the system (1) to a system with two independent variables.

The curves $W(\lambda, \mu)=0$ on which conditions for the reduced system must be prescribed are

I1a: $W(\lambda, \mu)=\mu$

I1b: $W(\lambda, \mu)=\lambda$

I1c: $W(\lambda, \mu)=\lambda-x_0$

I2a: $W(\lambda, \mu)=\lambda$

I2b: $W(\lambda, \mu)=\lambda-x_0$

II2: $W(\lambda, \mu)=\lambda-x_0$.

4.3. Boundary values on circles and ellipses

Let boundary values be prescribed on ellipses $(x-x_0)^2/a^2+(y-y_0)^2/b^2=1$, a, b, x_0, $y_0\in\mathbb{R}$, i.e. $w(x, y, t)=(x-x_0)^2/a^2+(y-y_0)^2/b^2-1$. The transformations which leave these curves invariant must satisfy

$$C_4=C_5=C_7=0 \qquad C_1=-C_6y_0 \qquad C_2=C_6x_0 \qquad C_6=0 \text{ or } a=b.$$

Hence cases I1c, I2b (circles and ellipses), and I1a, I2a, III1aα (circles only) can be used to reduce the system (1) to a system with two independent variables.

The curves $W(\lambda, \mu)=0$ on which conditions for the reduced system must be prescribed are

I1a: $W(\lambda, \mu)=\lambda^2+\mu^2-a^2$ $\quad(a=b$ only)

I1c: $W(\lambda, \mu)=\dfrac{(\lambda-x_0)^2}{a^2}+\dfrac{(\mu-y_0)^2}{b^2}-1$ $\quad(a\neq b$ possible)

I2a: $W(\lambda, \mu)=\lambda^2+\mu^2-a^2$ $\quad(a=b$ only)

I2b: $W(\lambda, \mu)=\dfrac{(\lambda-x_0)^2}{a^2}+\dfrac{(\mu-y_0)^2}{b^2}-1$ $\quad(a\neq b$ possible)

III1aα: $W(\lambda, \mu)=\lambda-\frac{1}{2}C_6\ln(a^2)$ $\quad(a=b$ only).

4.4. Boundary and initial values on a circle

Let boundary values be prescribed at the time t on the circle $w(x, y, t)=x^2+y^2-c^2t^2$, $c\in\mathbb{R}$. The transformations which leave this curve invariant must satisfy

$$C_1=C_2=C_3=C_4=C_5=0 \qquad C_7=C_8.$$

For $C_7=C_8\neq 0$ the case I1a ($C_6\neq 0$) or I1b ($C_6=0$) can be used to reduce the system (1) to a system with two independent variables.

The curves $W(\lambda, \mu)=0$ on which conditions for the reduced system must be prescribed are in both cases $W(\lambda, \mu)=\lambda^2+\mu^2-c^2$.

5. Reduction to ordinary differential equations

One can look once more for similarity solutions of any of the reduced systems of differential equations just obtained. These similarity solutions of the reduced system are also similarity solutions of the system (1), but in this way it is not necessary for all similarity solutions of (1) to be determined.

In order to obtain similarity solutions of the reduced system, the similarity variable and the new dependent variables which can be used to reduce the system of partial differential equations to a system of ordinary differential equations must be calculated. These calculations have been applied to the systems of cases I1c, I2b, II1aβ and II1b.

We again look for infinitesimal transformations which leave the given system of differential equations invariant. The associated infinitesimals are now ξ^λ, ξ^μ, η^U, η^V, η^H, η^R and η^P, depending on λ, μ, U, V, H, R and P. The new similarity variable is called Λ and the new dependent variables are called $\mathcal{U}(\Lambda)$, $\mathcal{V}(\Lambda)$, $\mathcal{H}(\Lambda)$, $\mathcal{R}(\Lambda)$ and $\mathcal{P}(\Lambda)$.

5.1. Reduction of the system corresponding to cases I1c and I2b

In cases I1c and I2b the systems of differential equations can be treated together, because they only differ in some constants. With the substitution

$$K_1 = (C_3C_4 - C_1C_8)/C_8^2 \qquad K_2 = (C_3C_5 - C_2C_8)/C_8^2$$
$$K_3 = C_9/C_8 \qquad K_4 = 2(C_8 + C_9)/C_8$$

in case I1c, or

$$K_1 = -C_1/C_3 \qquad K_2 = -C_2/C_3 \qquad K_3 = C_9/C_3 \qquad K_4 = 2C_9/C_3$$

in case I2b, both systems can be written as

$$\frac{\partial R}{\partial \lambda}(K_1 + U) + \frac{\partial R}{\partial \mu}(K_2 + V) + R\left(\frac{\partial U}{\partial \lambda} + \frac{\partial V}{\partial \mu}\right) + K_4 R = 0 \tag{10a}$$

$$R_H H\frac{\partial H}{\partial \lambda} + \frac{\partial P}{\partial \lambda} + R\frac{\partial U}{\partial \lambda}(K_1 + U) + R\frac{\partial U}{\partial \mu}(K_2 + V) - RU = 0 \tag{10b}$$

$$R_H H\frac{\partial H}{\partial \mu} + \frac{\partial P}{\partial \mu} + R\frac{\partial V}{\partial \lambda}(K_1 + U) + R\frac{\partial V}{\partial \mu}(K_2 + V) - RV = 0 \tag{10c}$$

$$\frac{\partial H}{\partial \lambda}(K_1 + U) + \frac{\partial H}{\partial \mu}(K_2 + V) + H\left(\frac{\partial U}{\partial \lambda} + \frac{\partial V}{\partial \mu}\right) + K_3 H = 0 \tag{10d}$$

$$\frac{\partial P}{\partial \lambda}(K_1 + U) + \frac{\partial P}{\partial \mu}(K_2 + V) + \gamma P\left(\frac{\partial U}{\partial \lambda} + \frac{\partial V}{\partial \mu}\right) + 2K_3 P = 0. \tag{10e}$$

The determining equations of the infinitesimals can be calculated. To solve these equations we make the ansatz that the infinitesimals are fourth degree polynomials in λ, μ, U, V, H, R and P. Hence, in general, the full solution of the determining equations is not derived. Instead of the most general transformation group, a subgroup may be obtained and the Lie algebra of the infinitesimal operators is a sub-algebra of the most general algebra.

Substituting the ansatz into the determining equations, a system of linear equations for the coefficients of the polynomials is obtained. The solution of this system contains three free constants $\tilde{C}_1, \tilde{C}_2, \tilde{C}_3 \in \mathbb{R}$ in the case of $\gamma \neq 2$ or $K_3 \neq K_4$ and five free constants $\tilde{C}_1, \ldots, \tilde{C}_5 \in \mathbb{R}$ in the case of $\gamma = 2$ and $K_3 = K_4$.

Finally we get infinitesimals in the two cases

(*a*) $\gamma \neq 2$ or $K_3 \neq K_4$:

$$\xi^\lambda = \tilde{C}_1 \qquad \xi^\mu = \tilde{C}_2 \qquad \eta^U = 0$$
$$\eta^V = 0 \qquad \eta^H = \tilde{C}_3 H \qquad \eta^R = 2\tilde{C}_3 R \qquad \eta^P = 2\tilde{C}_3 P \tag{11a}$$

(*b*) $\gamma=2$ and $K_3=K_4$:

$$\xi^\lambda=\tilde{C}_1 \qquad \xi^\mu=\tilde{C}_2 \qquad \eta^U=0$$
$$\eta^V=0 \qquad \eta^H=(\tilde{C}_3-\tilde{C}_4)H+\tilde{C}_5 R \qquad \eta^R=2\tilde{C}_3 R$$
$$\eta^P=2\tilde{C}_3 P+\tilde{C}_4 R_H H^2+\tilde{C}_5 R_H H R. \tag{11b}$$

The group for the case of $\gamma\neq 2$ or $K_3\neq K_4$ is a subgroup of the case of $\gamma=2$ and $K_3=K_4$, which can be obtained by setting $\tilde{C}_4=\tilde{C}_5=0$.

The set of the infinitesimal operators

$$Z=\xi^\lambda\frac{\partial}{\partial\lambda}+\xi^\mu\frac{\partial}{\partial\mu}+\eta^U\frac{\partial}{\partial U}+\eta^V\frac{\partial}{\partial V}+\eta^H\frac{\partial}{\partial H}+\eta^R\frac{\partial}{\partial R}+\eta^P\frac{\partial}{\partial P} \tag{12}$$

forms a three- or five-dimensional Lie algebra. In the case of $\gamma\neq 2$ or $K_3\neq K_4$ the Lie algebra is generated by

$$\tilde{X}_1=\partial/\partial\lambda \qquad \tilde{X}_2=\frac{\partial}{\partial\mu} \qquad \tilde{X}_3=H\frac{\partial}{\partial H}+2R\frac{\partial}{\partial R}+2P\frac{\partial}{\partial P}. \tag{13}$$

In the case of $\gamma=2$ and $K_3=K_4$ a basis of the Lie algebra is formed by $\tilde{X}_1$, $\tilde{X}_2$, $\tilde{X}_3$ and

$$\tilde{X}_4=-H\frac{\partial}{\partial H}+R_H H^2\frac{\partial}{\partial P} \qquad \tilde{X}_5=R\frac{\partial}{\partial H}-R_H H R\frac{\partial}{\partial P}. \tag{14}$$

The commutator table for this Lie algebra is given in the following table:

	$\tilde{X}_1$	$\tilde{X}_2$	$\tilde{X}_3$	$\tilde{X}_4$	$\tilde{X}_5$
$\tilde{X}_1$	0	0	0	0	0
$\tilde{X}_2$	0	0	0	0	0
$\tilde{X}_3$	0	0	0	0	$\tilde{X}_5$
$\tilde{X}_4$	0	0	0	0	$\tilde{X}_5$
$\tilde{X}_5$	0	0	$-\tilde{X}_5$	$-\tilde{X}_5$	0.

The global one-parameter transformations associated with these base vectors are

$\tilde{C}_1\neq 0$: $\tilde{\lambda}=\lambda+\varepsilon\tilde{C}_1$ (translation in the λ direction)

$\tilde{C}_2\neq 0$: $\tilde{\mu}=\mu+\varepsilon\tilde{C}_2$ (translation in the μ direction)

$\tilde{C}_3\neq 0$: $\tilde{H}=H\,e^{\tilde{C}_3\varepsilon}$ $\tilde{R}=R\,e^{2\tilde{C}_3\varepsilon}$ $\tilde{P}=P\,e^{2\tilde{C}_3\varepsilon}$
(scaling for H, R, P)

$\tilde{C}_4\neq 0$: $\tilde{H}=H\,e^{-\tilde{C}_4\varepsilon}$ $\tilde{P}=P-\frac{1}{2}R_H H^2(e^{-2\tilde{C}_4\varepsilon}-1)$

$\tilde{C}_5\neq 0$: $\tilde{H}=H-\varepsilon\tilde{C}_5 R$ $\tilde{P}=P+\varepsilon\tilde{C}_5 R_H H R-\frac{1}{2}\varepsilon^2\tilde{C}_5^2 R_H R^2$.

The new similarity variable Λ and the new dependent variables $\mathscr{U}$, $\mathscr{V}$, $\mathscr{H}$, $\mathscr{R}$ and $\mathscr{P}$ are obtained as constants by solving the system of differential equations

$$\frac{d\lambda}{\xi^\lambda}=\frac{d\mu}{\xi^\mu}=\frac{dU}{\eta^U}=\frac{dV}{\eta^V}=\frac{dH}{\eta^H}=\frac{dR}{\eta^R}=\frac{dP}{\eta^P}. \tag{15}$$

One obtains in the case of $\gamma\neq 2$ or $K_3\neq K_4$:

(*a*) $\tilde{C}_1\neq 0$:

$$\Lambda=\tilde{C}_2\lambda-\tilde{C}_1\mu \qquad U(\lambda,\mu)=\mathscr{U}(\Lambda) \qquad V(\lambda,\mu)=\mathscr{V}(\Lambda)$$
$$H(\lambda,\mu)=\mathscr{H}(\Lambda)\,e^{\tilde{C}_3\lambda/\tilde{C}_1} \qquad R(\lambda,\mu)=\mathscr{R}(\Lambda)\,e^{2\tilde{C}_3\lambda/\tilde{C}_1}$$
$$P(\lambda,\mu)=\mathscr{P}(\Lambda)\,e^{2\tilde{C}_3\lambda/\tilde{C}_1}. \tag{16}$$

The reduced system of differential equations is

$$0=\mathcal{R}(\mathcal{U}'\tilde{C}_2-\mathcal{V}'\tilde{C}_1)+\mathcal{R}'[\tilde{C}_2(\mathcal{U}+K_1)-\tilde{C}_1(\mathcal{V}+K_2)]+2(\tilde{C}_3/\tilde{C}_1)\mathcal{R}(\mathcal{U}+K_1)+\mathcal{R}K_4$$
$$0=\tilde{C}_2R_H\mathcal{H}\mathcal{H}'+\tilde{C}_2\mathcal{P}'-\mathcal{R}\mathcal{U}+\mathcal{R}\mathcal{U}'[\tilde{C}_2(\mathcal{U}+K_1)-\tilde{C}_1(\mathcal{V}+K_2)]+(\tilde{C}_3/\tilde{C}_1)(R_H\mathcal{H}^2+2\mathcal{P})$$
$$0=-\tilde{C}_1R_H\mathcal{H}\mathcal{H}'-\tilde{C}_1\mathcal{P}'-\mathcal{R}\mathcal{V}+\mathcal{R}\mathcal{V}'[\tilde{C}_2(\mathcal{U}+K_1)-\tilde{C}_1(\mathcal{V}+K_2)] \qquad (17)$$
$$0=\mathcal{H}(\mathcal{U}'\tilde{C}_2-\mathcal{V}'\tilde{C}_1)+\mathcal{H}'[\tilde{C}_2(\mathcal{U}+K_1)-\tilde{C}_1(\mathcal{V}+K_2)]+(\tilde{C}_3/\tilde{C}_1)\mathcal{H}(\mathcal{U}+K_1)+\mathcal{H}K_3$$
$$0=\gamma\mathcal{P}(\mathcal{U}'\tilde{C}_2-\mathcal{V}'\tilde{C}_1)+\mathcal{P}'[\tilde{C}_2(\mathcal{U}+K_1)-\tilde{C}_1(\mathcal{V}+K_2)]+2(\tilde{C}_3/\tilde{C}_1)\mathcal{P}(\mathcal{U}+K_1)+2\gamma\mathcal{P}K_3$$

where the prime denotes derivation with respect to Λ.

(*b*) $\tilde{C}_1=0$, $\tilde{C}_2\neq 0$:

$$\Lambda=\lambda \qquad U(\lambda,\mu)=\mathcal{U}(\Lambda) \qquad V(\lambda,\mu)=\mathcal{V}(\Lambda)$$
$$H(\lambda,\mu)=\mathcal{H}(\Lambda)\,e^{\tilde{C}_3\mu/\tilde{C}_2} \qquad R(\lambda,\mu)=\mathcal{R}(\Lambda)\,e^{2\tilde{C}_3\mu/\tilde{C}_2} \qquad (18)$$
$$P(\lambda,\mu)=\mathcal{P}(\Lambda)\,e^{2\tilde{C}_3\mu/\tilde{C}_2}.$$

The reduced system of differential equations is

$$0=\mathcal{R}\mathcal{U}'+2(\tilde{C}_3/\tilde{C}_2)\mathcal{R}(\mathcal{V}+K_2)+\mathcal{R}'(\mathcal{U}+K_1)+\mathcal{R}K_4$$
$$0=R_H\mathcal{H}\mathcal{H}'+\mathcal{P}'-\mathcal{R}\mathcal{U}+\mathcal{R}\mathcal{U}'(\mathcal{U}+K_1)$$
$$0=-\mathcal{R}\mathcal{V}+\mathcal{R}\mathcal{V}'(\mathcal{U}+K_1)+(\tilde{C}_3/\tilde{C}_2)(R_H\mathcal{H}^2+2\mathcal{P}) \qquad (19)$$
$$0=\mathcal{H}\mathcal{U}'+(\tilde{C}_3/\tilde{C}_2)\mathcal{H}(\mathcal{V}+K_2)+\mathcal{H}'(\mathcal{U}+K_1)+\mathcal{H}K_3$$
$$0=\gamma\mathcal{P}\mathcal{U}'+2(\tilde{C}_3/\tilde{C}_2)\mathcal{P}(\mathcal{V}+K_2)+\mathcal{P}'(\mathcal{U}+K_1)+2\gamma\mathcal{P}K_3.$$

5.2. Reduction of the system of case II1b

In the case under discussion we have calculated the infinitesimals in a way analogous to the preceding subsection. The result contains four free constants $\tilde{C}_1,\ldots,\tilde{C}_4\in\mathbb{R}$ and some different cases must be distinguished.

The infinitesimals are

(*a*) $\gamma\neq 1$, $\gamma\neq 2$ or $\gamma=1$; $C_1\neq C_4$ or $C_2\neq C_5$:

$$\xi^\lambda=\tilde{C}_1+\tilde{C}_2\mu \qquad \xi^\mu=0 \qquad \eta^U=0$$
$$\eta^V=\tilde{C}_2 \qquad \eta^H=0 \qquad \eta^R=0 \qquad \eta^P=0 \qquad (20a)$$

(*b*) $\gamma=1$, $C_1=C_4$, $C_2=C_5$:

$$\xi^\lambda=\tilde{C}_1+\tilde{C}_2\mu+\tilde{C}_4\lambda \qquad \xi^\mu=\tilde{C}_4+\tilde{C}_4\mu \qquad \eta^U=0$$
$$\eta^V=\tilde{C}_2 \qquad \eta^H=\tilde{C}_4H \qquad \eta^R=2\tilde{C}_4R \qquad \eta^P=2\tilde{C}_4P \qquad (20b)$$

(*c*) $\gamma=2$:

$$\xi^\lambda=\tilde{C}_1+\tilde{C}_2\mu \qquad \xi^\mu=0 \qquad \eta^U=0$$
$$\eta^V=\tilde{C}_2 \qquad \eta^H=0 \qquad \eta^R=2\tilde{C}_3R$$
$$\eta^P=2\tilde{C}_3P+R_H\tilde{C}_3H^2. \qquad (20c)$$

(In the case of $C_2=0$, $C_4=0$ or $C_9=0$ an additional solution may occur.)

Note that in case (*a*) the group of transformations is a subgroup both of the group in case (*b*) and of the group in case (*c*) which is obtained by setting $\tilde{C}_4=0$ or $\tilde{C}_3=0$. The Lie algebras of the associated infinitesimal operators are two or three dimensional, respectively, and their base vectors are

(*a*) $\gamma\neq 1$, $\gamma\neq 2$ or $\gamma=1$; $C_1\neq C_4$ or $C_2\neq C_5$:

$$\tilde{X}_1=\partial/\partial\lambda \qquad \tilde{X}_2=\mu\partial/\partial\lambda+\partial/\partial V \tag{21}$$

(*b*) $\gamma=1$, $C_1=C_4$, $C_2=C_5$:

$\tilde{X}_1$ and $\tilde{X}_2$ as in case (*a*),

$$\tilde{X}_4=\lambda\frac{\partial}{\partial\lambda}+(1+\mu)\frac{\partial}{\partial\mu}+H\frac{\partial}{\partial H}+2R\frac{\partial}{\partial R}+2P\frac{\partial}{\partial P} \tag{22}$$

(*c*) $\gamma=2$:

$\tilde{X}_1$ and $\tilde{X}_2$ as in case (*a*),

$$\tilde{X}_3=2R\frac{\partial}{\partial R}+(R_HH^2+2P)\frac{\partial}{\partial P}. \tag{23}$$

The Lie algebras of cases (*a*) and (*c*) are commutative Lie algebras; the commutator table of case (*b*) is given in the following table:

	$\tilde{X}_1$	$\tilde{X}_2$	$\tilde{X}_4$
$\tilde{X}_1$	0	0	$\tilde{X}_1$
$\tilde{X}_2$	0	0	$\tilde{X}_2$
$\tilde{X}_4$	$-\tilde{X}_1$	$-\tilde{X}_2$	0.

The global one-parameter transformations associated with the base vectors are

$\tilde{C}_1\neq 0$: $\tilde{\lambda}=\lambda+\varepsilon\tilde{C}_1$ (translation in the λ direction)

$\tilde{C}_2\neq 0$: $\tilde{\lambda}=\lambda+\varepsilon\tilde{C}_2\mu$ $\tilde{V}=V+\varepsilon\tilde{C}_2$

$\tilde{C}_3\neq 0$: $\tilde{R}=R\,e^{2\tilde{C}_3\varepsilon}$ $\tilde{P}=(P+\frac{1}{2}R_HH^2)\,e^{2\tilde{C}_3\varepsilon}-\frac{1}{2}R_HH^2$

$\tilde{C}_4\neq 0$: $\tilde{\lambda}=\lambda\,e^{\tilde{C}_4\varepsilon}$ $\tilde{\mu}=(1+\mu)\,e^{\tilde{C}_4\varepsilon}-1$

$\tilde{H}=H\,e^{\tilde{C}_4\varepsilon}$ $\tilde{R}=R\,e^{2\tilde{C}_4\varepsilon}$ $\tilde{P}=P\,e^{2\tilde{C}_4\varepsilon}$.

The similarity variable Λ, the new dependent variables $\mathcal{U}$, $\mathcal{V}$, $\mathcal{H}$, $\mathcal{R}$ and $\mathcal{P}$ and the reduced system of differential equations are as follows.

(*a*) $\gamma\neq 1$, $\gamma\neq 2$ or $\gamma=1$; $C_1\neq C_4$ or $C_2\neq C_5$:

$$\begin{gathered}\Lambda=\mu \qquad U(\lambda,\mu)=\mathcal{U}(\Lambda) \qquad V(\lambda,\mu)=\mathcal{V}(\Lambda)+\frac{\tilde{C}_2\lambda}{\tilde{C}_1+\tilde{C}_2\mu}\\ H(\lambda,\mu)=\mathcal{H}(\Lambda) \qquad R(\lambda,\mu)=\mathcal{R}(\Lambda) \qquad P(\lambda,\mu)=\mathcal{P}(\Lambda).\end{gathered} \tag{24}$$

The reduced system of differential equations is

$$\begin{aligned}
0&=\mathcal{R}'(C_4\Lambda+C_1)(\tilde{C}_2\Lambda+\tilde{C}_1)+\mathcal{R}[2\mathcal{U}C_9(\tilde{C}_2\Lambda+\tilde{C}_1)+\tilde{C}_2(C_4\Lambda+C_1)+C_4(\tilde{C}_2\Lambda+\tilde{C}_1)]\\
0&=\mathcal{U}'\mathcal{R}(C_4\Lambda+C_1)+\mathcal{U}\mathcal{R}C_4+C_9R_H\mathcal{H}^2+2C_9\mathcal{P}\\
0&=\mathcal{V}'(C_4\Lambda+C_1)(\tilde{C}_2\Lambda+\tilde{C}_1)+\mathcal{U}(C_5\tilde{C}_1-C_2\tilde{C}_2)+\mathcal{V}\tilde{C}_2(C_4\Lambda+C_1)\\
0&=\mathcal{H}'(C_4\Lambda+C_1)(\tilde{C}_2\Lambda+\tilde{C}_1)+\mathcal{H}[\mathcal{U}C_9(\tilde{C}_2\Lambda+\tilde{C}_1)+\tilde{C}_2(C_4\Lambda+C_1)+C_4(\tilde{C}_2\Lambda+\tilde{C}_1)]\\
0&=\mathcal{P}'(C_4\Lambda+C_1)(\tilde{C}_2\Lambda+\tilde{C}_1)+\mathcal{P}[2\mathcal{U}C_9(\tilde{C}_2\Lambda+\tilde{C}_1)+\gamma\tilde{C}_2(C_4\Lambda+C_1)+\gamma C_4(\tilde{C}_2\Lambda+\tilde{C}_1)].
\end{aligned} \tag{25}$$

(*b*) $\gamma=1$, $C_1=C_4$, $C_2=C_5$ ($\tilde{C}_4\neq 0$, in the case of $\tilde{C}_4=0$ the same solution as in case (*a*) is obtained):

$$\begin{aligned} &\Lambda=\frac{\lambda}{1+\mu}+\frac{\tilde{C}_1-\tilde{C}_2}{\tilde{C}_4(1+\mu)}-\frac{\tilde{C}_2}{\tilde{C}_4}\ln(1+\mu) \qquad U(\lambda,\mu)=\mathscr{U}(\Lambda)\\ &V(\lambda,\mu)=\mathscr{V}(\Lambda)+(\tilde{C}_2/\tilde{C}_4)\ln(1+\mu) \qquad H(\lambda,\mu)=\mathscr{H}(\Lambda)(1+\mu)\\ &R(\lambda,\mu)=\mathscr{R}(\Lambda)(1+\mu)^2 \qquad P(\lambda,\mu)=\mathscr{P}(\Lambda)(1+\mu)^2. \end{aligned} \tag{26}$$

The reduced system of differential equations is

$$\begin{aligned} 0&=\mathscr{R}'(C_4\tilde{C}_4\mathscr{V}-C_5\tilde{C}_4\mathscr{U}-C_4\tilde{C}_4\Lambda-C_4\tilde{C}_2)+\mathscr{R}\tilde{C}_4(C_4\mathscr{V}'-C_5\mathscr{U}'+2C_9\mathscr{U}+3C_4)\\ 0&=\mathscr{R}\mathscr{U}'(C_4\tilde{C}_4\mathscr{V}-C_5\tilde{C}_4\mathscr{U}-C_4\tilde{C}_4\Lambda-C_4\tilde{C}_2)+C_4\tilde{C}_4\mathscr{R}\mathscr{U}-C_5\tilde{C}_4R_H\mathscr{H}\mathscr{H}'\\ &\quad-C_5\tilde{C}_4\mathscr{P}'+C_9\tilde{C}_4R_H\mathscr{H}^2+2C_9\tilde{C}_4\mathscr{P}\\ 0&=\mathscr{R}\mathscr{V}'(C_4\tilde{C}_4\mathscr{V}-C_5\tilde{C}_4\mathscr{U}-C_4\tilde{C}_4\Lambda-C_4\tilde{C}_2)+C_5\tilde{C}_4\mathscr{R}\mathscr{U}+C_4\tilde{C}_4R_H\mathscr{H}\mathscr{H}'\\ &\quad+C_4\tilde{C}_4\mathscr{P}'+C_4\tilde{C}_2\mathscr{R}\\ 0&=\mathscr{H}'(C_4\tilde{C}_4\mathscr{V}-C_5\tilde{C}_4\mathscr{U}-C_4\tilde{C}_4\Lambda-C_4\tilde{C}_2)+\mathscr{H}\tilde{C}_4(C_4\mathscr{V}'-C_5\mathscr{U}'+C_9\mathscr{U}+2C_4)\\ 0&=\mathscr{P}'(C_4\tilde{C}_4\mathscr{V}-C_5\tilde{C}_4\mathscr{U}-C_4\tilde{C}_4\Lambda-C_4\tilde{C}_2)+\mathscr{P}\tilde{C}_4(C_4\mathscr{V}'-C_5\mathscr{U}'+2C_9\mathscr{U}+3C_4). \end{aligned} \tag{27}$$

(*c*) $\gamma=2$ ($\tilde{C}_3\neq 0$, in the case of $\tilde{C}_3=0$ the same solution as in case (*a*) is obtained):

$$\begin{aligned} &\Lambda=\mu \qquad U(\lambda,\mu)=\mathscr{U}(\Lambda)\\ &V(\lambda,\mu)=\mathscr{V}(\Lambda)+\frac{\tilde{C}_2\lambda}{\tilde{C}_1+\tilde{C}_2\mu} \qquad H(\lambda,\mu)=\mathscr{H}(\Lambda)\\ &R(\lambda,\mu)=\mathscr{R}(\Lambda)\exp\left(\frac{2\tilde{C}_3\lambda}{\tilde{C}_1+\tilde{C}_2\mu}\right)\\ &P(\lambda,\mu)=\mathscr{P}(\Lambda)\exp\left(\frac{2\tilde{C}_3\lambda}{\tilde{C}_1+\tilde{C}_2\mu}\right)-\tfrac{1}{2}R_H\mathscr{H}(\Lambda). \end{aligned} \tag{28}$$

The reduced system of differential equations is

$$\begin{aligned} 0&=\mathscr{R}'(C_4\Lambda+C_1)(\tilde{C}_2\Lambda+\tilde{C}_1)+\mathscr{R}[2C_9\mathscr{U}(\tilde{C}_2\Lambda+\tilde{C}_1)-2\tilde{C}_3\mathscr{U}(C_5\Lambda+C_2)\\ &\quad+2\tilde{C}_3\mathscr{V}(C_4\Lambda+C_1)+\tilde{C}_2(C_4\Lambda+C_1)+C_4(\tilde{C}_2\Lambda+\tilde{C}_1)]\\ 0&=\mathscr{R}\mathscr{U}'(C_4\Lambda+C_1)(\tilde{C}_2\Lambda+\tilde{C}_1)+\mathscr{R}\mathscr{U}C_4(\tilde{C}_2\Lambda+\tilde{C}_1)\\ &\quad+2\mathscr{P}[C_9(\tilde{C}_2\Lambda+\tilde{C}_1)-\tilde{C}_3(C_5\Lambda+C_2)]\\ 0&=\mathscr{R}\mathscr{V}'(C_4\Lambda+C_1)(\tilde{C}_2\Lambda+\tilde{C}_1)+\mathscr{R}\mathscr{V}\tilde{C}_2(C_4\Lambda+C_1)+\mathscr{R}\mathscr{U}(C_5\tilde{C}_1-C_2\tilde{C}_2)\\ &\quad+2\mathscr{P}\tilde{C}_3(C_4\Lambda+C_1)\\ 0&=\mathscr{H}'(C_4\Lambda+C_1)(\tilde{C}_2\Lambda+\tilde{C}_1)+\mathscr{H}[C_9\mathscr{U}(\tilde{C}_2\Lambda+\tilde{C}_1)+\tilde{C}_2(C_4\Lambda+C_1)\\ &\quad+C_4(\tilde{C}_2\Lambda+\tilde{C}_1)]\\ 0&=\mathscr{P}'(C_4\Lambda+C_1)(\tilde{C}_2\Lambda+\tilde{C}_1)+\mathscr{P}[2C_9\mathscr{U}(\tilde{C}_2\Lambda+\tilde{C}_1)-2\tilde{C}_3\mathscr{U}(C_5\Lambda+C_2)\\ &\quad+2\tilde{C}_3\mathscr{V}(C_4\Lambda+C_1)+2\tilde{C}_2(C_4\Lambda+C_1)+2C_4(\tilde{C}_2\Lambda+\tilde{C}_1)]. \end{aligned} \tag{29}$$

5.3. *Reduction of the system of case IIIaβ*

The calculations of the infinitesimals, the similarity variable, the new dependent variables and the reduced system correspond to the previous cases. Here only the results are given.

The infinitesimals are

$$\xi^{\lambda} = \tilde{C}_1 \qquad \tilde{C}_1 \in \mathbb{R} \qquad \xi^{\mu} = 0 \qquad \eta^{U} = 0$$
$$\eta^{V} = 0 \qquad \eta^{H} = 0 \qquad \eta^{P} = 0. \tag{30}$$

The Lie algebra of the infinitesimal operators is one dimensional, its base vector being

$$\tilde{X}_1 = \partial/\partial\lambda. \tag{31}$$

The corresponding global one-parameter transformation is a translation in the λ direction:

$$\tilde{\lambda} = \lambda + \varepsilon\tilde{C}_1.$$

The similarity variable and the new dependent variables are

$$\Lambda = \mu \qquad U(\lambda, \mu) = \mathcal{U}(\Lambda) \qquad V(\lambda, \mu) = \mathcal{V}(\Lambda)$$
$$H(\lambda, \mu) = \mathcal{H}(\Lambda) \qquad R(\lambda, \mu) = \mathcal{R}(\Lambda) \qquad P(\lambda, \mu) = \mathcal{P}(\Lambda). \tag{32}$$

The reduced system of differential equations is

$$\begin{aligned}
0 &= \mathcal{R}'(\mathcal{V} - \Lambda\mathcal{U}) + \mathcal{R}(\mathcal{V}' - \Lambda\mathcal{U}') + \mathcal{R}\mathcal{U}(2C_9/C_7 - 1) \\
0 &= \mathcal{R}\mathcal{U}'(\mathcal{V} - \Lambda\mathcal{U}) + \mathcal{R}(\mathcal{U})^2 - C_7^2 R_H \Lambda\mathcal{H}\mathcal{H}' - C_7^2\Lambda\mathcal{P}' + C_7C_9R_H\mathcal{H}^2 + 2C_7C_9\mathcal{P} \\
0 &= \mathcal{R}\mathcal{V}'(\mathcal{V} - \Lambda\mathcal{U}) + \mathcal{R}\mathcal{U}\mathcal{V} + C_7^2R_H\mathcal{H}\mathcal{H}' + C_7^2\mathcal{P}' \\
0 &= \mathcal{H}'(\mathcal{V} - \Lambda\mathcal{U}) + \mathcal{H}(\mathcal{V}' - \Lambda\mathcal{U}') + \mathcal{H}\mathcal{U}(C_9/C_7 + 1) \\
0 &= \mathcal{P}'(\mathcal{V} - \Lambda\mathcal{U}) + \gamma\mathcal{P}(\mathcal{V}' - \Lambda\mathcal{U}') + \mathcal{P}\mathcal{U}(2C_9/C_7 + \gamma).
\end{aligned} \tag{33}$$

All the systems of ordinary differential equations (17), (19), (25), (27), (29) and (33) are non-linear. Therefore a separate qualitative and numerical investigtion is needed to analyse the corresponding dynamic systems. This is beyond the scope of our present work.

Finally, as an example we illustrate the return to the original variables for (33) only. If a solution $\mathcal{U}$, $\mathcal{V}$, $\mathcal{H}$, $\mathcal{R}$ and $\mathcal{P}$ of (33) is calculated for particular initial conditions $\mathcal{U}(\Lambda_0)$, $\mathcal{V}(\Lambda_0)$, $\mathcal{H}(\Lambda_0)$, $\mathcal{R}(\Lambda_0)$ and $\mathcal{P}(\Lambda_0)$, the solution of (1) is obtained by going back to the original variables with (32) and the corresponding equations for case IIIaβ given in the appendix. This leads to the following similarity solution of (1):

$$\begin{aligned}
v^x(x, y, t) &= \frac{C_7x + C_4t + C_1}{C_7}\mathcal{U}(\Lambda) - \frac{C_4}{C_7} \\
v^y(x, y, t) &= \frac{C_7x + C_4t + C_1}{C_7}\mathcal{V}(\Lambda) - \frac{C_5}{C_7} \\
h(x, y, t) &= (C_7x + C_4t + C_1)^{C_9/C_7}\mathcal{H}(\Lambda) \\
\rho(x, y, t) &= (C_7x + C_4t + C_1)^{2(C_9 - C_7)/C_7}\mathcal{R}(\Lambda) \\
p(x, y, t) &= (C_7x + C_4t + C_1)^{2C_9/C_7}\mathcal{P}(\Lambda)
\end{aligned} \tag{34}$$

with

$$\Lambda = (C_7y + C_5t + C_2)/(C_7x + C_4t + C_1).$$

6. Concluding remarks

The full Lie symmetry group admitted by the hyperbolic system of partial differential equations (1) has been found. As an important subgroup the group of scalings (also termed 'dilatations' or 'stretchings') should be mentioned. In the present case it is at most a three-parameter group, given by $C_7 \neq 0$, $C_8 \neq 0$ and $C_9 \neq 0$ in (3). Solutions which are invariant under this subgroup are called self-similar. They include solutions which are obtained by dimensional analysis. In many cases self-similar solutions are important for describing the asymptotic ($t \to \infty$) behaviour of non-stationary flows (Liberman and Velikovich 1986).

Due to the special structure of (1) the theory of this system is very similar to that of ordinary compressible gas dynamics (Grad 1960). Therefore it is interesting to compare the base elements of the two Lie algebras. The calculation of Lie symmetry groups of the motion of a perfect polytropic gas in n dimensions by Ibragimov (1985) yields a larger group in the case of $\gamma = (n+2)/n$ than in the case of $\gamma \neq (n+2)/n$. In agreement with this result for $n = 2$, we found that (1) admits a larger group in the case of $\gamma = 2$ than in the case of $\gamma \neq 2$. If we compare the two Lie algebras, we find that the elements X_1 up to X_8 of (5) correspond completely to operators given by Ibragimov. On the other hand, X_9 and X_{10} are equivalent to operators given by Ibragimov only in the case $h = 0$ and an operator such as $X(f)$ does not exist there.

It is interesting to remark that a larger group in the case of $\gamma = 2$ may occur in non-two-dimensional problems. For example, for the one-dimensional non-stationary real MHD equations with a density power law for the electrical conductivity, Groß (1983) obtained a larger group in the case of $\gamma = 2$, while according to Ibragimov one expects the larger one in the case of $\gamma = 3$. Obviously the terms with electrical conductivity change the structure of the system of differential equations, so that the formula of Ibragimov is no longer applicable. In particular, the real MHD equations are a system of second-order differential equations, whereas the ideal MHD equations are a system of first-order differential equations.

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Appendix

In what follows we summarise results in terms of the different cases discussed in § 4. The representation is suitable for practical reasons. For all possible values of the parameters the corresponding similarity systems can be found in this summary (except for $C_9 \neq 0$ only, see § 4). For any particular case one has to examine which parameters are not equal to zero and then look for the corresponding case in the following summary. We give some examples to illustrate this procedure.

Rotation and spatial translations ($C_1 \neq 0$, $C_2 \neq 0$, $C_6 \neq 0$ only): case II1aα.

Galilei transformations and time translations ($C_3 \neq 0$, $C_4 \neq 0$, $C_5 \neq 0$ only): case I2b.

Scalings (only if $C_7 \neq 0$, $C_9 \neq 0$): case II1aβ

(only if $C_8 \neq 0$, $C_9 \neq 0$): case I1c

(only if $C_7 \neq 0$, $C_8 \neq 0$, $C_9 \neq 0$): cases I1a and I1b.

Thus for the last example two cases are left. But only one similarity system exists since for $C_4 = C_5 = 0$ in case I1a $C_7 = C_8$ is allowed and the systems of I1a and I1b are equivalent.

Case I: $C_3 \neq 0$ or $C_8 \neq 0$

Case I1: $C_8 \neq 0$

Case I1a: $C_6 \neq 0$ or $C_6 = 0$, $C_7 \neq 0$, $C_7 \neq C_8$

$$\lambda = \left[(x - B_1)\left(t + \frac{C_3}{C_8}\right)^{-C_7/C_8} - A_1\left(t + \frac{C_3}{C_8}\right)^{-C_7/C_8+1}\right]\sin(g(t))$$
$$+\left[(y - B_2)\left(t + \frac{C_3}{C_8}\right)^{-C_7/C_8} - A_2\left(t + \frac{C_3}{C_8}\right)^{-C_7/C_8+1}\right]\cos(g(t))$$

$$\mu = \left[(x - B_1)\left(t + \frac{C_3}{C_8}\right)^{-C_7/C_8} - A_1\left(t + \frac{C_3}{C_8}\right)^{-C_7/C_8+1}\right]\cos(g(t))$$
$$-\left[(y - B_2)\left(t + \frac{C_3}{C_8}\right)^{-C_7/C_8} - A_2\left(t + \frac{C_3}{C_8}\right)^{-C_7/C_8+1}\right]\sin(g(t))$$

$$v^x(x, y, t) = (t + C_3/C_8)^{C_7/C_8 - 1}[U(\lambda, \mu)\cos(g(t)) + V(\lambda, \mu)\sin(g(t))] + A_1$$

$$v^y(x, y, t) = (t + C_3/C_8)^{C_7/C_8 - 1}[-U(\lambda, \mu)\sin(g(t)) + V(\lambda, \mu)\cos(g(t))] + A_2$$

$$h(x, y, t) = H(\lambda, \mu)(t + C_3/C_8)^{C_9/C_8}$$

$$\rho(x, y, t) = R(\lambda, \mu)(t + C_3/C_8)^{2(C_9 + C_8 - C_7)/C_8}$$

$$p(x, y, t) = P(\lambda, \mu)(t + C_3/C_8)^{2C_9/C_8}$$

with

$$A_1 = \frac{C_5C_6 - C_4(C_7 - C_8)}{C_6^2 + (C_7 - C_8)^2} \qquad B_1 = \frac{C_6(C_2C_8 - C_3C_5) - C_7(C_1C_8 - C_3C_4)}{C_8(C_6^2 + C_7^2)}$$

$$A_2 = -\frac{C_4C_6 + C_5(C_7 - C_8)}{C_6^2 + (C_7 - C_8)^2} \qquad B_2 = -\frac{C_6(C_1C_8 - C_3C_4) + C_7(C_2C_8 - C_3C_5)}{C_8(C_6^2 + C_7^2)}$$

$$g(t) = \frac{C_6}{C_8}\ln\left(t + \frac{C_3}{C_8}\right)$$

$$0 = \frac{\partial R}{\partial \lambda}\left(-\frac{C_7}{C_8}\lambda - \frac{C_6}{C_8}\mu + U\right) + \frac{\partial R}{\partial \mu}\left(\frac{C_6}{C_8}\lambda - \frac{C_7}{C_8}\mu + V\right) + R\left(\frac{\partial U}{\partial \lambda} + \frac{\partial V}{\partial \mu}\right)$$
$$+ R\frac{2(C_9 + C_8 - C_7)}{C_8}$$

$$0 = R_H H\frac{\partial H}{\partial \lambda} + \frac{\partial P}{\partial \lambda} + R\frac{\partial U}{\partial \lambda}\left(-\frac{C_7}{C_8}\lambda - \frac{C_6}{C_8}\mu + U\right) + R\frac{\partial U}{\partial \mu}\left(\frac{C_6}{C_8}\lambda - \frac{C_7}{C_8}\mu + V\right)$$
$$+\left(\frac{C_7}{C_8} - 1\right)RU + \frac{C_6}{C_8}RV$$

$$0 = R_H H\frac{\partial H}{\partial \mu} + \frac{\partial P}{\partial \mu} + R\frac{\partial V}{\partial \lambda}\left(-\frac{C_7}{C_8}\lambda - \frac{C_6}{C_8}\mu + U\right) + R\frac{\partial V}{\partial \mu}\left(\frac{C_6}{C_8}\lambda - \frac{C_7}{C_8}\mu + V\right)$$
$$-\frac{C_6}{C_8}RU + \left(\frac{C_7}{C_8} - 1\right)RV$$

$$0=\frac{\partial H}{\partial\lambda}\left(-\frac{C_7}{C_8}\lambda-\frac{C_6}{C_8}\mu+U\right)+\frac{\partial H}{\partial\mu}\left(\frac{C_6}{C_8}\lambda-\frac{C_7}{C_8}\mu+V\right)+H\left(\frac{\partial U}{\partial\lambda}+\frac{\partial V}{\partial\mu}\right)+H\frac{C_9}{C_8}$$

$$0=\frac{\partial P}{\partial\lambda}\left(-\frac{C_7}{C_8}\lambda-\frac{C_6}{C_8}\mu+U\right)+\frac{\partial P}{\partial\mu}\left(\frac{C_6}{C_8}\lambda-\frac{C_7}{C_8}\mu+V\right)+\gamma P\left(\frac{\partial U}{\partial\lambda}+\frac{\partial V}{\partial\mu}\right)+2P\frac{C_9}{C_8}.$$

Case I1b: $C_6=0,\ C_7=C_8\neq 0$

$$\lambda=\left(x+\frac{C_1C_8-C_3C_4}{C_8^2}\right)\left(t+\frac{C_3}{C_8}\right)^{-1}-\frac{C_4}{C_8}\ln\left(t+\frac{C_3}{C_8}\right)$$

$$\mu=\left(y+\frac{C_2C_8-C_3C_5}{C_8^2}\right)\left(t+\frac{C_3}{C_8}\right)^{-1}-\frac{C_5}{C_8}\ln\left(t+\frac{C_3}{C_8}\right)$$

$$v^x=U(\lambda,\mu)+\frac{C_4}{C_8}\ln\left(t+\frac{C_3}{C_8}\right)\qquad v^y=V(\lambda,\mu)+\frac{C_5}{C_8}\ln\left(t+\frac{C_3}{C_8}\right)$$

$$h=H(\lambda,\mu)(t+C_3/C_8)^{C_9/C_8}\qquad \rho=R(\lambda,\mu)(t+C_3/C_8)^{2C_9/C_8}$$

$$p=P(\lambda,\mu)(t+C_3/C_8)^{2C_9/C_8}$$

$$0=\frac{\partial R}{\partial\lambda}\left(-\frac{C_4}{C_8}-\lambda+U\right)+\frac{\partial R}{\partial\mu}\left(-\frac{C_5}{C_8}-\mu+V\right)+R\left(\frac{\partial U}{\partial\lambda}+\frac{\partial V}{\partial\mu}\right)+2R\frac{C_9}{C_8}$$

$$0=R_HH\frac{\partial H}{\partial\lambda}+\frac{\partial P}{\partial\lambda}+R\frac{\partial U}{\partial\lambda}\left(-\frac{C_4}{C_8}-\lambda+U\right)+R\frac{\partial U}{\partial\mu}\left(-\frac{C_5}{C_8}-\mu+V\right)+R\frac{C_4}{C_8}$$

$$0=R_HH\frac{\partial H}{\partial\mu}+\frac{\partial P}{\partial\mu}+R\frac{\partial V}{\partial\lambda}\left(-\frac{C_4}{C_8}-\lambda+U\right)+R\frac{\partial V}{\partial\mu}\left(-\frac{C_5}{C_8}-\mu+V\right)+R\frac{C_5}{C_8}$$

$$0=\frac{\partial H}{\partial\lambda}\left(-\frac{C_4}{C_8}-\lambda+U\right)+\frac{\partial H}{\partial\mu}\left(-\frac{C_5}{C_8}-\mu+V\right)+H\left(\frac{\partial U}{\partial\lambda}+\frac{\partial V}{\partial\mu}\right)+H\frac{C_9}{C_8}$$

$$0=\frac{\partial P}{\partial\lambda}\left(-\frac{C_4}{C_8}-\lambda+U\right)+\frac{\partial P}{\partial\mu}\left(-\frac{C_5}{C_8}-\mu+V\right)+\gamma P\left(\frac{\partial U}{\partial\lambda}+\frac{\partial V}{\partial\mu}\right)+2P\frac{C_9}{C_8}.$$

Case I1c: $C_6=C_7=0$

$$\lambda=x-\frac{C_4}{C_8}t-\frac{C_1C_8-C_3C_4}{C_8^2}\ln\left(t+\frac{C_3}{C_8}\right)$$

$$\mu=y-\frac{C_5}{C_8}t-\frac{C_2C_8-C_3C_5}{C_8^2}\ln\left(t+\frac{C_3}{C_8}\right)$$

$$v^x(x,y,t)=U(\lambda,\mu)(t+C_3/C_8)^{-1}+C_4/C_8$$

$$v^y(x,y,t)=V(\lambda,\mu)(t+C_3/C_8)^{-1}+C_5/C_8$$

$$h(x,y,t)=H(\lambda,\mu)(t+C_3/C_8)^{C_9/C_8}$$

$$\rho(x,y,t)=R(\lambda,\mu)(t+C_3/C_8)^{2(C_9+C_8)/C_8}$$

$$p(x,y,t)=P(\lambda,\mu)(t+C_3/C_8)^{2C_9/C_8}$$

$$0=\frac{\partial R}{\partial\lambda}\left(\frac{C_3C_4}{C_8^2}-\frac{C_1}{C_8}+U\right)+\frac{\partial R}{\partial\mu}\left(\frac{C_3C_5}{C_8^2}-\frac{C_2}{C_8}+V\right)+R\left(\frac{\partial U}{\partial\lambda}+\frac{\partial V}{\partial\mu}\right)+2R\frac{C_8+C_9}{C_8}$$

$$0=R_HH\frac{\partial H}{\partial\lambda}+\frac{\partial P}{\partial\lambda}+R\frac{\partial U}{\partial\lambda}\left(\frac{C_3C_4}{C_8^2}-\frac{C_1}{C_8}+U\right)+R\frac{\partial U}{\partial\mu}\left(\frac{C_3C_5}{C_8^2}-\frac{C_2}{C_8}+V\right)-RU$$

$$0=R_HH\frac{\partial H}{\partial\mu}+\frac{\partial P}{\partial\mu}+R\frac{\partial V}{\partial\lambda}\left(\frac{C_3C_4}{C_8^2}-\frac{C_1}{C_8}+U\right)+R\frac{\partial V}{\partial\mu}\left(\frac{C_3C_5}{C_8^2}-\frac{C_2}{C_8}+V\right)-RV$$

$$0=\frac{\partial H}{\partial\lambda}\left(\frac{C_3C_4}{C_8^2}-\frac{C_1}{C_8}+U\right)+\frac{\partial H}{\partial\mu}\left(\frac{C_3C_5}{C_8^2}-\frac{C_2}{C_8}+V\right)+H\left(\frac{\partial U}{\partial\lambda}+\frac{\partial V}{\partial\mu}\right)+H\frac{C_9}{C_8}$$

$$0=\frac{\partial P}{\partial\lambda}\left(\frac{C_3C_4}{C_8^2}-\frac{C_1}{C_8}+U\right)+\frac{\partial P}{\partial\mu}\left(\frac{C_3C_5}{C_8^2}-\frac{C_2}{C_8}+V\right)+\gamma P\left(\frac{\partial U}{\partial\lambda}+\frac{\partial V}{\partial\mu}\right)+2P\frac{C_9}{C_8}.$$

Case I2: $C_8=0,\ C_3\neq 0$

Case I2a: $C_6\neq 0$ or $C_7\neq 0$

$$\lambda=\mathrm{e}^{-C_7t/C_3}[(x-A_1-B_1t)\cos(C_6t/C_3)-(y-A_2-B_2t)\sin(C_6t/C_3)]$$

$$\mu=\mathrm{e}^{-C_7t/C_3}[(x-A_1-B_1t)\sin(C_6t/C_3)+(y-A_2-B_2t)\cos(C_6t/C_3)]$$

$$v^x(x,y,t)=\mathrm{e}^{C_7t/C_3}[U(\lambda,\mu)\cos(C_6t/C_3)+V(\lambda,\mu)\sin(C_6t/C_3)]+B_1$$

$$v^y(x,y,t)=\mathrm{e}^{C_7t/C_3}[-U(\lambda,\mu)\sin(C_6t/C_3)+V(\lambda,\mu)\cos(C_6t/C_3)]+B_2$$

$$h(x,y,t)=H(\lambda,\mu)\,\mathrm{e}^{C_9t/C_3}$$

$$\rho(x,y,t)=R(\lambda,\mu)\,\mathrm{e}^{2(C_9-C_7)t/C_3}$$

$$p(x,y,t)=P(\lambda,\mu)\,\mathrm{e}^{2C_9t/C_3}$$

with

$$A_1=\frac{2C_3C_5C_6C_7-C_3C_4(C_7^2-C_6^2)-(C_1C_7-C_2C_6)(C_7^2+C_6^2)}{(C_6^2+C_7^2)^2}$$

$$A_2=-\frac{2C_3C_4C_6C_7+C_3C_5(C_7^2-C_6^2)+(C_1C_6+C_2C_7)(C_7^2+C_6^2)}{(C_7^2+C_6^2)^2}$$

$$B_1=\frac{-C_4C_7+C_5C_6}{C_6^2+C_7^2}\qquad B_2=-\frac{C_4C_6+C_5C_7}{C_6^2+C_7^2}$$

$$0=\frac{\partial R}{\partial\lambda}\left(-\frac{C_7}{C_3}\lambda-\frac{C_6}{C_3}\mu+U\right)+\frac{\partial R}{\partial\mu}\left(\frac{C_6}{C_3}\lambda-\frac{C_7}{C_3}\mu+V\right)+R\left(\frac{\partial U}{\partial\lambda}+\frac{\partial V}{\partial\mu}\right)+R\frac{2(C_9-C_6)}{C_3}$$

$$0=R_HH\frac{\partial H}{\partial\lambda}+\frac{\partial P}{\partial\lambda}+R\frac{\partial U}{\partial\lambda}\left(-\frac{C_7}{C_3}\lambda-\frac{C_6}{C_3}\mu+U\right)+R\frac{\partial U}{\partial\mu}\left(\frac{C_6}{C_3}\lambda-\frac{C_7}{C_3}\mu+V\right)$$
$$+\frac{C_7}{C_3}RU+\frac{C_6}{C_3}RV$$

$$0=R_HH\frac{\partial H}{\partial\mu}+\frac{\partial P}{\partial\mu}+R\frac{\partial V}{\partial\lambda}\left(-\frac{C_7}{C_3}\lambda-\frac{C_6}{C_3}\mu+U\right)+R\frac{\partial V}{\partial\mu}\left(\frac{C_6}{C_3}\lambda-\frac{C_7}{C_3}\mu+V\right)$$
$$-\frac{C_6}{C_3}RU+\frac{C_7}{C_3}RV$$

$$0=\frac{\partial H}{\partial\lambda}\left(-\frac{C_7}{C_3}\lambda-\frac{C_6}{C_3}\mu+U\right)+\frac{\partial H}{\partial\mu}\left(\frac{C_6}{C_3}\lambda-\frac{C_7}{C_3}\mu+V\right)+H\left(\frac{\partial U}{\partial\lambda}+\frac{\partial V}{\partial\mu}\right)+H\frac{C_9}{C_3}$$

$$0=\frac{\partial P}{\partial\lambda}\left(-\frac{C_7}{C_3}\lambda-\frac{C_6}{C_3}\mu+U\right)+\frac{\partial P}{\partial\mu}\left(\frac{C_6}{C_3}\lambda-\frac{C_7}{C_3}\mu+V\right)+\gamma P\left(\frac{\partial U}{\partial\lambda}+\frac{\partial V}{\partial\mu}\right)+2P\frac{C_9}{C_3}.$$

Case I2b: $C_6=C_7=0$

$$\lambda=x-\frac{C_4}{2C_3}t^2-\frac{C_1}{C_3}t\qquad \mu=y-\frac{C_5}{2C_3}t^2-\frac{C_2}{C_3}t$$

$$v^x(x,y,t)=U(\lambda,\mu)+C_4t/C_3\qquad v^y(x,y,t)=V(\lambda,\mu)+C_5t/C_3$$

$$h(x,y,t)=H(\lambda,\mu)\,\mathrm{e}^{C_9t/C_3}\qquad \rho(x,y,t)=R(\lambda,\mu)\,\mathrm{e}^{2C_9t/C_3}$$

$$p(x,y,t)=P(\lambda,\mu)\,\mathrm{e}^{2C_9t/C_3}$$

$$0=\frac{\partial R}{\partial\lambda}\left(-\frac{C_1}{C_3}+U\right)+\frac{\partial R}{\partial\mu}\left(-\frac{C_2}{C_3}+V\right)+R\left(\frac{\partial U}{\partial\lambda}+\frac{\partial V}{\partial\mu}\right)+2R\frac{C_9}{C_3}$$

$$0=R_H H\frac{\partial H}{\partial\lambda}+\frac{\partial P}{\partial\lambda}+R\frac{\partial U}{\partial\lambda}\left(-\frac{C_1}{C_3}+U\right)+R\frac{\partial U}{\partial\mu}\left(-\frac{C_2}{C_3}+V\right)+R\frac{C_4}{C_3}$$

$$0=R_H H\frac{\partial H}{\partial\mu}+\frac{\partial P}{\partial\mu}+R\frac{\partial V}{\partial\lambda}\left(-\frac{C_1}{C_3}+U\right)+R\frac{\partial V}{\partial\mu}\left(-\frac{C_2}{C_3}+V\right)+R\frac{C_5}{C_3}$$

$$0=\frac{\partial H}{\partial\lambda}\left(-\frac{C_1}{C_3}+U\right)+\frac{\partial H}{\partial\mu}\left(-\frac{C_2}{C_3}+V\right)+H\left(\frac{\partial U}{\partial\lambda}+\frac{\partial V}{\partial\mu}\right)+H\frac{C_9}{C_3}$$

$$0=\frac{\partial P}{\partial\lambda}\left(-\frac{C_1}{C_3}+U\right)+\frac{\partial P}{\partial\mu}\left(-\frac{C_2}{C_3}+V\right)+\gamma P\left(\frac{\partial U}{\partial\lambda}+\frac{\partial V}{\partial\mu}\right)+2P\frac{C_9}{C_3}.$$

Case II: $C_3=C_8=0$

Case II1: $C_1\neq 0$ or $C_4\neq 0$ or $C_6\neq 0$ or $C_7\neq 0$

Case II1a: $C_6\neq 0$ or $C_7\neq 0$

Case II1aα: $C_6\neq 0$

$$\lambda=t \qquad \mu=C_7\tan^{-1}\frac{y-y_0(t)}{x-x_0(t)}+\tfrac{1}{2}C_6\ln[(x-x_0(t))^2+(y-y_0(t))^2]$$

$$v^x(x,y,t)=\frac{\exp\{-(C_7/C_6)\tan^{-1}[(y-y_0(t))/(x-x_0(t))]\}}{[(x-x_0(t))^2+(y-y_0(t))^2]^{1/2}}$$
$$\times[U(\lambda,\mu)(x-x_0(t))-V(\lambda,\mu)(y-y_0(t))]+\frac{C_5C_6-C_4C_7}{C_6^2+C_7^2}$$

$$v^y(x,y,t)=\frac{\exp\{-(C_7/C_6)\tan^{-1}[(y-y_0(t))/(x-x_0(t))]\}}{[(x-x_0(t))^2+(y-y_0(t))^2]^{1/2}}$$
$$\times[V(\lambda,\mu)(x-x_0(t))+U(\lambda,\mu)(y-y_0(t))]-\frac{C_4C_6+C_5C_7}{C_6^2+C_7^2}$$

$$h(x,y,t)=H(\lambda,\mu)\exp\left(-\frac{C_9}{C_6}\tan^{-1}\frac{y-y_0(t)}{x-x_0(t)}\right)$$

$$\rho(x,y,t)=R(\lambda,\mu)\exp\left(-\frac{2(C_9-C_7)}{C_6}\tan^{-1}\frac{y-y_0(t)}{x-x_0(t)}\right)$$

$$p(x,y,t)=P(\lambda,\mu)\exp\left(-\frac{2C_9}{C_6}\tan^{-1}\frac{y-y_0(t)}{x-x_0(t)}\right)$$

with

$$x_0(t)=\frac{(C_6C_5-C_4C_7)t+C_2C_6-C_1C_7}{C_6^2+C_7^2}$$

$$y_0(t)=-\frac{(C_5C_7+C_4C_6)t+C_2C_7+C_1C_6}{C_6^2+C_7^2}$$

$$0=\frac{\partial R}{\partial\mu}(C_6U+C_7V)+R\left(C_6\frac{\partial U}{\partial\mu}+C_7\frac{\partial V}{\partial\mu}\right)+RU+\frac{C_7-2C_9}{C_6}RV+\mathrm{e}^{\mu/C_6}\frac{\partial R}{\partial\lambda}$$

$$0=C_7R_HH\frac{\partial H}{\partial\mu}+C_7\frac{\partial P}{\partial\mu}+R\frac{\partial V}{\partial\mu}(C_7V+C_6U)+R\,\mathrm{e}^{\mu/C_6}\frac{\partial V}{\partial\lambda}+RV\left(U-\frac{C_7}{C_6}V\right)$$
$$-\frac{C_9}{C_6}R_HH^2-2\frac{C_9}{C_6}P$$

$$0=C_6R_HH\frac{\partial H}{\partial\mu}+C_6\frac{\partial P}{\partial\mu}+R\frac{\partial U}{\partial\mu}(C_7V+C_6U)+R\,\mathrm{e}^{\mu/C_6}\frac{\partial U}{\partial\lambda}+RV\left(-\frac{C_7}{C_6}U-V\right)$$

$$0=\frac{\partial H}{\partial\mu}(C_6U+C_7V)+H\left(C_6\frac{\partial U}{\partial\mu}+C_7\frac{\partial V}{\partial\mu}\right)+HU-\frac{C_7+C_9}{C_6}HV+\mathrm{e}^{\mu/C_6}\frac{\partial H}{\partial\lambda}$$

$$0=\frac{\partial P}{\partial\mu}(C_6U+C_7V)+\gamma P\left(C_6\frac{\partial U}{\partial\mu}+C_7\frac{\partial V}{\partial\mu}\right)+\gamma PU-\frac{\gamma C_7+2C_9}{C_6}PV+\mathrm{e}^{\mu/C_6}\frac{\partial P}{\partial\lambda}.$$

Case II1aβ: $C_6=0$, $C_7\neq 0$

$$\lambda=t \qquad \mu=\frac{C_7y+C_5t+C_2}{C_7x+C_4t+C_1}$$

$$v^x(x,y,t)=\frac{C_7x+C_4t+C_1}{C_7}U(\lambda,\mu)-\frac{C_4}{C_7}$$

$$v^y(x,y,t)=\frac{C_7x+C_4t+C_1}{C_7}V(\lambda,\mu)-\frac{C_5}{C_7}$$

$$h(x,y,t)=H(\lambda,\mu)(C_7x+C_4t+C_1)^{C_9/C_7}$$

$$\rho(x,y,t)=R(\lambda,\mu)(C_7x+C_4t+C_1)^{2(C_9-C_7)/C_7}$$

$$p(x,y,t)=P(\lambda,\mu)(C_7x+C_4t+C_1)^{2C_9/C_7}$$

$$0=\frac{\partial R}{\partial\lambda}+\frac{\partial R}{\partial\mu}(V-\mu U)+R\left(\frac{\partial V}{\partial\mu}-\frac{\partial U}{\partial\mu}\mu\right)+RU\left(2\frac{C_9}{C_7}-1\right)$$

$$0=\frac{\partial U}{\partial\lambda}R+\frac{\partial U}{\partial\mu}R(V-\mu U)+RU^2-C_7^2R_H\mu H\frac{\partial H}{\partial\mu}-C_7^2\mu\frac{\partial P}{\partial\mu}+C_9C_7R_HH^2+2C_9C_7P$$

$$0=\frac{\partial V}{\partial\lambda}R+\frac{\partial V}{\partial\mu}R(V-\mu U)+RUV+C_7^2R_HH\frac{\partial H}{\partial\mu}+C_7^2\frac{\partial P}{\partial\mu}$$

$$0=\frac{\partial H}{\partial\lambda}+\frac{\partial H}{\partial\mu}(V-\mu U)+H\left(\frac{\partial V}{\partial\mu}-\frac{\partial U}{\partial\mu}\mu\right)+HU\left(\frac{C_9}{C_7}+1\right)$$

$$0=\frac{\partial P}{\partial\lambda}+\frac{\partial P}{\partial\mu}(V-\mu U)+\gamma P\left(\frac{\partial V}{\partial\mu}-\frac{\partial U}{\partial\mu}\mu\right)+PU\left(2\frac{C_9}{C_7}+\gamma\right).$$

Case II1b: $C_6=C_7=0$, $C_1\neq 0$ or $C_4\neq 0$

$$\lambda=t \qquad \mu=y-\frac{C_5t+C_2}{C_4t+C_1}x$$

$$v^x(x,y,t)=U(\lambda,\mu)+\frac{C_4x}{C_4t+C_1}$$

$$v^y(x,y,t)=V(\lambda,\mu)+\frac{C_5x}{C_4t+C_1}$$

$$h(x,y,t)=H(\lambda,\mu)\exp\left(\frac{C_9x}{C_4t+C_1}\right)$$

$$\rho(x,y,t)=R(\lambda,\mu)\exp\left(\frac{2C_9x}{C_4t+C_1}\right)$$

$$p(x,y,t)=P(\lambda,\mu)\exp\left(\frac{2C_9x}{C_4t+C_1}\right)$$

$$0=\frac{\partial R}{\partial \mu}[U(-C_5\lambda-C_2)+V(C_4\lambda+C_1)]+\frac{\partial R}{\partial \lambda}(C_4\lambda+C_1)$$
$$+R\left(\frac{\partial U}{\partial \mu}(-C_5\lambda-C_2)+\frac{\partial V}{\partial \mu}(C_4\lambda+C_1)\right)+R(2C_9U+C_4)$$

$$0=R_HH\frac{\partial H}{\partial \mu}(-C_5\lambda-C_2)+\frac{\partial P}{\partial \mu}(-C_5\lambda-C_2)+R\frac{\partial U}{\partial \lambda}(C_4\lambda+C_1)$$
$$+R\frac{\partial U}{\partial \mu}[U(-C_5\lambda-C_2)+V(C_4\lambda+C_1)]+C_4RU+C_9(R_HH^2+2P)$$

$$0=R_HH\frac{\partial H}{\partial \mu}(C_4\lambda+C_1)+\frac{\partial P}{\partial \mu}(C_4\lambda+C_1)+R\frac{\partial V}{\partial \lambda}(C_4\lambda+C_1)$$
$$+R\frac{\partial V}{\partial \mu}[U(-C_5\lambda-C_2)+V(C_4\lambda+C_1)]+C_5RU$$

$$0=\frac{\partial H}{\partial \mu}[U(-C_5\lambda-C_2)+V(C_4\lambda+C_1)]+\frac{\partial H}{\partial \lambda}(C_4\lambda+C_1)$$
$$+H\left(\frac{\partial U}{\partial \mu}(-C_5\lambda-C_2)+\frac{\partial V}{\partial \mu}(C_4\lambda+C_1)\right)+H(C_9U+C_4)$$

$$0=\frac{\partial P}{\partial \mu}[U(-C_5\lambda-C_2)+V(C_4\lambda+C_1)]+\frac{\partial P}{\partial \lambda}(C_4\lambda+C_1)$$
$$+\gamma P\left(\frac{\partial U}{\partial \mu}(-C_5\lambda-C_2)+\frac{\partial V}{\partial \mu}(C_4\lambda+C_1)\right)+P(2C_9U+\gamma C_4).$$

Case II2: $C_1=C_4=C_6=C_7=0$, $C_2\neq 0$ or $C_5\neq 0$

$$\lambda=t \qquad \mu=x$$
$$v^x(x,y,t)=U(\lambda,\mu)$$
$$v^y(x,y,t)=V(\lambda,\mu)+\frac{C_5y}{C_5t+C_2}$$
$$h(x,y,t)=H(\lambda,\mu)\exp\left(\frac{C_9y}{C_5t+C_2}\right)$$
$$\rho(x,y,t)=R(\lambda,\mu)\exp\left(\frac{2C_9y}{C_5t+C_2}\right)$$
$$p(x,y,t)=P(\lambda,\mu)\exp\left(\frac{2C_9y}{C_5t+C_2}\right)$$

$$0=\frac{\partial R}{\partial \lambda}(C_5\lambda+C_2)+\frac{\partial R}{\partial \mu}U(C_5\lambda+C_2)+R\frac{\partial U}{\partial \mu}(C_5\lambda+C_2)+R(2C_9V+C_5)$$

$$0=R_HH\frac{\partial H}{\partial \mu}+\frac{\partial P}{\partial \mu}+RU\frac{\partial U}{\partial \mu}+R\frac{\partial U}{\partial \lambda}$$

$$0=RU\frac{\partial V}{\partial \mu}(C_5\lambda+C_2)+R\frac{\partial V}{\partial \lambda}(C_5\lambda+C_2)+C_5RV+C_9(R_HH^2+P)$$

$$0=\frac{\partial H}{\partial \lambda}(C_5\lambda+C_2)+\frac{\partial H}{\partial \mu}U(C_5\lambda+C_2)+H\frac{\partial U}{\partial \mu}(C_5\lambda+C_2)+H(C_9V+C_5)$$

$$0=\frac{\partial P}{\partial \lambda}(C_5\lambda+C_2)+\frac{\partial P}{\partial \mu}U(C_5\lambda+C_2)+\gamma P\frac{\partial U}{\partial \mu}(C_5\lambda+C_2)+P(2C_9V+\gamma C_5).$$

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